IX.

NOTE ON THE AUTOMORPHIC LINEAR TRANSFORMATION OF A BILINEAR FORM.

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§ 1.

IN the *Bulletin of the New York Mathematical Society* for July, 1894, I have shown that not every proper orthogonal substitution can be generated by the repetition of an infinitesimal orthogonal substitution. That is to say, if we designate a substitution of the orthogonal group as of the first or second kind according as it is or is not the second power of a substitution of the group, there are then proper orthogonal substitutions of the second kind; and whereas any substitution of the first kind can be generated by the repetition of an infinitesimal substitution of the group, no substitution of the second kind can be generated thus. Nevertheless, by the repetition of an infinitesimal substitution of the orthogonal group, we can obtain a substitution of the first kind which shall be as nearly as we please equal to any proper substitution whatever of the second kind.

I also pointed out in this paper that for the orthogonal group every substitution of the first kind was the mth power of a substitution of the group, for any positive integer m; and that every substitution of the second kind was the $(2\,m + 1)$th power of a substitution of the group.

It follows of course at once that an exactly similar theory exists for the group of linear substitutions which transform automorphically a symmetric bilinear form with cogrediant variables. An exactly similar theory exists also for the group of linear substitutions which transform automorphically an alternate bilinear form with cogrediant variables, and for the group of linear substitutions which transform automorphically a general bilinear form (neither symmetric nor alternate) with cogrediant variables, as remarked in a note at the conclusion of the above mentioned article.

On the other hand, any linear substitution of the group of linear substitutions which transform automorphically a bilinear form with

contragrediant variables can be generated by the repetition of an infinitesimal substitution of the group. For, if the two sets of variables of the bilinear form

$$(\Omega \between x_1, x_2, \ldots x_n \between y_1, y_2, \ldots y_n)$$

are contragrediant, and if

$$(x_1, x_2, \ldots x_n) = (\phi \between \xi_1, \xi_2, \ldots \xi_n),$$

$$(\eta_1, \eta_2, \ldots \eta_n) = (\breve{\phi} \between y_1, y_2, \ldots y_n),$$

in which $\breve{\phi}$ denotes the transverse or conjugate to ϕ, we have

$$(\phi^{-1} \Omega \phi \between \xi_1, \xi_2, \ldots \xi_n \between \eta_1, \eta_2, \ldots \eta_n)$$
$$= (\Omega \between x_1, x_2, \ldots x_n \between y_1, y_2, \ldots y_n);$$

and the necessary and sufficient condition that the transformation shall be automorphic is

$$\phi^{-1} \Omega \phi = \Omega,$$

or

$$\Omega \phi = \phi \Omega.*$$

Let now δ denote the identical substitution. Since it is assumed that a reciprocal of ϕ exists, that is, $|\phi| \neq 0$, a polynomial $\chi = f(\phi)$ in ϕ can be found such that

$$\phi = e^{\chi},$$

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* In this paper I employ the notation of Cayley's "Memoir on the Linear Automorphic Transformation of a Bipartite Quadric Function," *Philosophical Transactions*, 1858, with these exceptions, namely, the identical substitution will be denoted by δ, and the linear substitution or matrix transverse or conjugate to the linear substitution or matrix ϕ will be denoted by $\breve{\phi}$. Cayley denotes the bilinear form $\Sigma_r \Sigma_s a_{rs} x_s y_r$ $(r, s = 1, 2, \ldots n)$, as above, by

$$(\Omega \between x_1, x_2, \ldots x_n \between y_1, y_2, \ldots y_n),$$

the symbol Ω denoting the matrix

$$\begin{matrix} a_{11} \, a_{12} \cdots \\ a_{21} \, a_{22} \cdots \\ \cdots\cdots \end{matrix}$$

that is, the square array of coefficients of the form.

The determinant of the linear substitution ϕ will be denoted by $|\phi|$.

where e^{χ} denotes the convergent infinite series

$$\delta + \chi + \frac{1}{2}\chi^2 + \frac{1}{3\,!}\chi^3 + \cdots + \frac{1}{m\,!}\chi^m + \cdots *$$

Since ϕ is commutative with Ω, χ is also commutative with Ω; consequently for any positive integer m, $\frac{1}{m}\chi$ and therefore $e^{\frac{1}{m}\chi}$ are commutative with Ω. Whence it follows that if

$$\psi = e^{\frac{1}{m}\chi},$$

we have

$$\psi^{-1}\,\Omega\,\psi = \Omega,$$

and

$$\psi^m = e^{\chi} = \phi.$$

That is, any linear substitution which transforms automorphically the bilinear form

$$(\Omega \between x_1, x_2, \ldots x_n \between y_1, y_2, \ldots y_n)$$

with contragrediant variables is the mth power of a linear substitution which also transforms this form automorphically. By taking m sufficiently great, the coefficients of the linear substitution χ can be made as nearly as we please equal to zero, and thus the linear substitution $\psi = e^{\frac{1}{m}\chi}$ may be made as nearly as we please equal to the identical substitution. But however great m may be, we have, nevertheless,

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* The infinite series

$$e^{\chi} = \delta + \chi + \frac{1}{2}\chi^2 + \frac{1}{3!}\chi^3 + \cdots + \frac{1}{m\,!}\chi^m + \cdots$$

is convergent for any linear substitution χ; and we have

$$\left(e^{\chi}\right)^{-1} = e^{-\chi},$$

$$\breve{\left(e^{\chi}\right)} = e^{\breve{\chi}},$$

and if m is any positive integer,

$$\left(e^{\chi}\right)^m = e^{m\chi}.$$

If χ and χ' are commutative, we also have

$$e^{\chi}\,e^{\chi'} = e^{\chi+\chi'}.$$

For any linear substitution ϕ whose determinant is not zero a polynomial $\chi = f(\phi)$ in ϕ can be found such that

$$\phi = e^{\chi}.$$

$$\psi^{-1}\,\Omega\,\psi = \Omega,$$
$$\psi^{m} = \phi\,;$$

whence it follows that any linear substitution of the group of linear substitutions which transform automorphically a bilinear form with contragrediant variables can be generated by the repetition of an infinitesimal substitution of the group.

§ 2.

If the two sets of variables of the bilinear form

$$(\Omega \between x_1,\, x_2,\, \ldots\, x_n \between y_1,\, y_2,\, \ldots\, y_n)$$

are transformed by linear substitutions transverse or conjugate to each other, so that

$$(x_1,\, x_2,\, \ldots\, x_n) = (\phi \between \xi_1,\, \xi_2,\, \ldots\, \xi_n),$$
$$(y_1,\, y_2,\, \ldots\, y_n) = (\breve{\phi} \between \eta_1,\, \eta_2,\, \ldots\, \eta_n),$$

where $\breve{\phi}$ denotes the linear substitution transverse or conjugate to ϕ, we have

$$(\phi\,\Omega\,\phi \between \xi_1,\, \xi_2,\, \ldots\, \xi_n \between \eta_1,\, \eta_2,\, \ldots\, \eta_n)$$
$$= (\Omega \between x_1,\, x_2,\, \ldots\, x_n \between y_1,\, y_2,\, \ldots\, y_n).$$

The necessary and sufficient condition that this transformation shall be automorphic is that ϕ shall satisfy the equation

$$\phi\,\Omega\,\phi = \Omega.$$

The class of linear substitutions that satisfy this equation, that is, the linear substitutions which transform the bilinear form in the manner described, do not form a group; but they can be separated into substitutions of the first or second kind according as they are or are not the second power of a substitution of this class.* And any substitution of the first kind can then be generated by the repetition of an infinitesimal substitution of this class, whereas no substitution of the second kind can be generated thus.

* If $\phi\,\Omega\,\phi = \Omega$, then $\phi^2\,\Omega\,\phi^2 = \phi\,(\phi\,\Omega\,\phi)\,\phi = \phi\,\Omega\,\phi = \Omega$.

For assuming that the determinant of the form is not zero, that is, $|\Omega| \neq 0$, it follows that $|\phi| \neq 0$. Consequently a polynomial $\chi = f(\phi)$ can be found such that

$$\phi = e^{\chi}.*$$

Let

$$\vartheta = f(\phi)\,\Omega^{-1},$$

then

$$\phi = e^{\vartheta\,\Omega}.$$

From the identity

$$e^{-\Omega\,\vartheta}\,\Omega\,e^{\vartheta\,\Omega} = \Omega\,\dagger$$

we have also

$$\phi = e^{-\Omega\vartheta} = \left(e^{\Omega\,\vartheta}\right)^{-1}.$$

Since

$$\phi^{-1} = \Omega\,\phi\,\Omega^{-1},$$

therefore

$$\Omega\,\vartheta = \Omega\,(\vartheta\,\Omega)\,\Omega^{-1} = \Omega f(\phi)\,\Omega^{-1} = f(\Omega\,\phi\,\Omega^{-1}) = f(\phi^{-1})\,\ddagger$$

is a polynomial in ϕ, and consequently commutative with $\vartheta\,\Omega$. Whence we have *

$$e^{\vartheta\,\Omega + \Omega\,\vartheta} = e^{\vartheta\,\Omega}\,e^{\Omega\,\vartheta} = \phi\,\phi^{-1} = \delta.$$

Conversely, if $\vartheta\,\Omega$ and $\Omega\,\vartheta$ are commutative and such that

$$e^{\vartheta\,\Omega + \Omega\,\vartheta} = \delta,$$

then if

$$\phi = e^{\vartheta\,\Omega},$$

from the preceding identity it follows that

$$\phi\,\Omega\,\phi = \Omega.$$

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* See note, page 183.

† For any positive integer m we have

$$\Omega\,(\vartheta\,\Omega)^m = (\Omega\,\vartheta)^m\,\Omega.$$

Therefore

$$e^{-\Omega\,\vartheta}\,.\,\Omega\,e^{\vartheta\,\Omega} = e^{-\Omega\,\vartheta}\,.\,e^{\Omega\,\vartheta}\,.\,\Omega = \Omega.$$

‡ Since

$$f(\phi) = \Sigma_m\,c_m\,\phi^m,$$

$$\Omega f(\phi)\,\Omega^{-1} = \Omega\,(\Sigma_m\,c_m\,\phi^m)\,\Omega^{-1} = \Sigma_m\,c_m\,\Omega\,\phi^m\,\Omega^{-1} = \Sigma_m\,c_m\,(\Omega\,\phi\,\Omega^{-1})^m$$
$$= \Sigma_m\,c_m\,(\phi^{-1})^m = f(\phi^{-1}).$$

In particular, if $\vartheta\,\Omega = -\,\Omega\,\vartheta$, $\phi = e^{\vartheta\,\Omega}$ satisfies the preceding equation.

Let now

$$\theta_0\,\Omega = \tfrac{1}{2}\,(\vartheta\,\Omega + \Omega\,\vartheta) = \tfrac{1}{2}\,\big(f(\phi) + f(\phi^{-1})\big),$$

then

$$\begin{aligned}\Omega\,\theta_0 &= \tfrac{1}{2}\,\Omega\,(\vartheta\,\Omega + \Omega\,\vartheta)\,\Omega^{-1}\\ &= \tfrac{1}{2}\,\big(\Omega f(\phi)\,\Omega^{-1} + \Omega f(\phi^{-1})\,\Omega^{-1}\big)\\ &= \tfrac{1}{2}\,\big(f(\Omega\,\phi\,\Omega^{-1}) + f(\Omega\,\phi^{-1}\,\Omega^{-1})\big)\\ &= \tfrac{1}{2}\,\big(f(\phi^{-1}) + f(\phi)\big) = \theta_0\,\Omega.^*\end{aligned}$$

Again, let

$$\theta_1\,\Omega = \tfrac{1}{2}\,(\vartheta\,\Omega - \Omega\,\vartheta) = \tfrac{1}{2}\,\big(f(\phi) - f(\phi^{-1})\big)\,;$$

then

$$\begin{aligned}\Omega\,\theta_1 &= \tfrac{1}{2}\,\Omega\,\big(f(\phi) - f(\phi^{-1})\big)\,\Omega^{-1}\\ &= \tfrac{1}{2}\,\big(\Omega f(\phi)\,\Omega^{-1} - \Omega f(\phi^{-1})\,\Omega^{-1}\big)\\ &= \tfrac{1}{2}\,\big(f(\Omega\,\phi\,\Omega^{-1}) - f(\Omega\,\phi^{-1}\,\Omega^{-1})\big)\\ &= \tfrac{1}{2}\,\big(f(\phi^{-1}) - f(\phi)\big) = -\,\theta_1\,\Omega.\end{aligned}$$

Since $\theta_0\,\Omega$ and $\theta_1\,\Omega$ are polynomials in ϕ, they are commutative. Therefore

$$\phi^2 = e^{2\,\vartheta\,\Omega} = e^{2\,\theta_0\,\Omega + 2\,\theta_1\,\Omega} = e^{2\,\theta_0\,\Omega}\,e^{2\,\theta_1\,\Omega} = e^{2\,\theta_1\,\Omega},$$

since

$$e^{2\,\theta_0\,\Omega} = e^{\vartheta\,\Omega + \Omega\,\vartheta} = \delta.$$

If now

$$\psi = e^{\frac{2}{m}\,\theta_1\,\Omega} = e^{-\,\frac{2}{m}\,\Omega\,\theta_1},$$

$$\psi\,\Omega\,\psi = e^{-\frac{2}{m}\,\Omega\,\theta_1}\,\Omega\,e^{\frac{2}{m}\,\theta_1\,\Omega} = \Omega,$$

and

$$\psi^m = \Big(e^{\frac{2}{m}\,\theta_1\,\Omega}\Big)^m = e^{2\,\theta_1\,\Omega} = \phi^2.$$

Consequently, any linear substitution, as ϕ^2, which is the second power of a linear substitution satisfying the equation

$$\phi\,\Omega\,\phi = \Omega,$$

is the mth power for any positive integer m of a solution of this equation. By taking m sufficiently great, we can make the coefficients

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* See last note, page 185.

of $\frac{2}{m}\,\theta_1\,\Omega$ as small as we please, and thus we may make the substitution ψ as nearly as we please equal to the identical substitution. Whence it follows that any linear substitution of the first kind which satisfies the equation

$$\phi\,\Omega\,\phi = \Omega$$

(that is, any linear substitution which is the second power of a solution of this equation) can be generated by the repetition of a linear substitution which is also a solution of this equation and which is infinitely near the identical substitution.

Any linear substitution ϕ satisfying the equation

$$\phi\,\Omega\,\phi = \Omega$$

is of the first kind if -1 is not a root of the characteristic equation of ϕ (that is, ϕ is then the second power of a substitution satisfying this equation). For if -1 is not a root of the characteristic equation of ϕ, we may put

$$\Omega^{-1}\,\Upsilon = -\,\delta + 2\,(\delta + \phi)^{-1} = (\delta - \phi)\,(\delta + \phi)^{-1};$$

and we then have

$$\begin{aligned}
\Upsilon\,\Omega^{-1} &= \Omega\,(\Omega^{-1}\,\Upsilon)\,\Omega^{-1} \\
&= \Omega\,(\delta - \phi)\,(\delta + \phi)^{-1}\,\Omega^{-1} \\
&= \Omega\,(\delta - \phi)\,\Omega^{-1}\cdot\Omega\,(\delta + \phi)^{-1}\,\Omega^{-1} \\
&= (\delta - \Omega\,\phi\,\Omega^{-1})\,(\delta + \Omega\,\phi\,\Omega^{-1})^{-1} \\
&= (\delta - \phi^{-1})\,(\delta + \phi^{-1})^{-1} \\
&= (\phi - \delta)\,(\phi + \delta)^{-1} = -\,\Omega^{-1}\,\Upsilon.
\end{aligned}$$

From the expression for $\Omega^{-1}\,\Upsilon$ we also obtain

$$(\Omega^{-1}\,\Upsilon + \delta)\,(\phi + \delta) = 2\,\delta;$$

and consequently, since $|\,\Omega^{-1}\,\Upsilon + \delta\,| \neq 0$,

$$\phi = -\,\delta + 2\,(\delta + \Omega^{-1}\Upsilon)^{-1} = (\delta + \Omega^{-1}\,\Upsilon)^{-1}\,(\delta - \Omega^{-1}\Upsilon).$$

If now $\vartheta = f\,(\Omega^{-1}\,\Upsilon)$ is a polynomial in $\Omega^{-1}\,\Upsilon$ such that

$$\delta + \Omega^{-1}\,\Upsilon = e^{\vartheta},$$

then, if $\vartheta' = f(-\Omega^{-1}\Upsilon)$ we have

$$\delta - \Omega^{-1}\Upsilon = e^{\vartheta'};*$$

and consequently

$$\phi = (e^{\vartheta})^{-1} e^{\vartheta'} = e^{-\vartheta+\vartheta'}.$$

Wherefore, if

$$\theta\Omega = -\vartheta + \vartheta' = -f(\Omega^{-1}\Upsilon) + f(-\Omega^{-1}\Upsilon),$$

$$\begin{aligned}\Omega\theta &= \Omega\left(-f(\Omega^{-1}\Upsilon) + f(-\Omega^{-1}\Upsilon)\right)\Omega^{-1}\\ &= -\Omega f(\Omega^{-1}\Upsilon)\Omega^{-1} + \Omega f(-\Omega^{-1}\Upsilon)\Omega^{-1}\\ &= -f(\Upsilon\Omega^{-1}) + f(-\Upsilon\Omega^{-1})\\ &= -f(-\Omega^{-1}\Upsilon) + f(\Omega^{-1}\Upsilon) = -\theta\Omega;\end{aligned}$$

and consequently, if

$$\psi = e^{\frac{1}{2}\theta\Omega} = e^{-\frac{1}{2}\Omega\theta},$$

we have

$$\psi\,\Omega\,\psi = \Omega,$$

and

$$\psi^2 = e^{\theta\Omega} = e^{-\vartheta+\vartheta'} = \phi;$$

that is, ϕ is the second power of a solution of the equation

$$\phi\,\Omega\,\phi = \Omega.$$

If a linear substitution satisfying this equation is sufficiently near to the identical substitution, —1 is not a root of its characteristic equation. Therefore an infinitesimal substitution satisfying this equation is of the first kind. But the repetition of a substitution of the first kind gives a substitution of that kind. Whence it follows that no substitution of the second kind, which satisfies the equation, can be

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* For

$$\begin{aligned}\vartheta' &= f(-\Omega^{-1}\Upsilon)\\ &= f(\Upsilon\Omega^{-1})\\ &= f(\Omega\cdot\Omega^{-1}\Upsilon\cdot\Omega^{-1})\\ &= \Omega f(\Omega^{-1}\Upsilon)\Omega^{-1} = \Omega\vartheta\Omega^{-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}e^{\vartheta'} &= e^{\Omega\vartheta\Omega^{-1}}\\ &= \Omega e^{\vartheta}\Omega^{-1}\\ &= \Omega(\delta + \Omega^{-1}\Upsilon)\Omega^{-1}\\ &= \delta + \Upsilon\Omega^{-1} = \delta - \Omega^{-1}\Upsilon.\end{aligned}$$

generated by the repetition of an infinitesimal substitution satisfying this equation.

Let ϕ be a linear substitution of the first kind of the class we are now considering. That is, let

$$\psi\,\Omega\,\psi = \Omega,$$

and let

$$\phi = \psi^2.$$

The roots of the characteristic equation of ϕ are then the squares of the roots of the characteristic equation of ψ. Consequently, if -1 is a root of the characteristic equation of ϕ, $\sqrt{-1}$ is a root of the characteristic equation of ψ; that is,

$$|\,\psi - \sqrt{-1}\,\delta\,| = 0.$$

But then

$$|\,\psi^{-1} - \sqrt{-1}\,\delta\,| = |\,\Omega\,(\psi^{-1} - \sqrt{-1}\,\delta)\,\Omega^{-1}\,| = |\,\psi - \sqrt{-1}\,\delta\,| = 0;$$

and since

$$\psi^{-1} - \sqrt{-1}\,\delta = -\sqrt{-1}\,\psi^{-1}\,(\psi + \sqrt{-1}\,\delta),$$

we have

$$|\,\psi + \sqrt{-1}\,\delta\,| = 0;$$

that is, $-\sqrt{-1}$ is then also a root of the characteristic equation of ψ. It is convenient at this point to introduce a term which has been employed by Sylvester. Thus, following Sylvester, I shall say that the *nullity* of the linear substitution Φ is m, if all the $(m-1)$th minors of the matrix or determinant of Φ are zero, (that is, the minors of order $n - m + 1$, if n is the number of variables,) but not all the mth minors (the minors of order $n-m$). If now the nullity of $\psi - \sqrt{-1}\,\delta$ is m, then, since

$$\psi - \sqrt{-1}\,\delta = \Omega\,(\psi^{-1} - \sqrt{-1}\,\delta)\,\Omega^{-1},$$

the nullity of

$$-\sqrt{-1}\,\psi^{-1}\,(\psi + \sqrt{-1}\,\delta) = \psi^{-1} - \sqrt{-1}\,\delta$$

is also m; therefore, since $|\,\psi^{-1}\,| \neq 0$, the nullity of $\psi + \sqrt{-1}\,\delta$ is m. Whence, by the "corollary of the law of nullity," the nullity of

$$\phi + \delta = (\psi - \sqrt{-1}\,\delta)\,(\psi + \sqrt{-1}\,\delta)$$

is then $2m$. Similarly, if p is any positive integer, and if the nullity of

$$(\psi - \sqrt{-1}\,\delta)^p = \Omega\,(\psi^{-1} - \sqrt{-1}\,\delta)^p\,\Omega^{-1}$$

is m, the nullity of

$$(-\sqrt{-1}\,\psi^{-1})^p\,(\psi + \sqrt{-1}\,\delta)^p = (\psi^{-1} - \sqrt{-1}\,\delta)^p$$

is also m; and therefore the nullity of $(\psi + \sqrt{-1}\,\delta)^p$ is m. But then the nullity of

$$(\phi + \delta)^p = (\psi - \sqrt{-1}\,\delta)^p\,(\psi + \sqrt{-1}\,\delta)^p$$

is $2m$.

We have therefore the following theorem by which we may ascertain the existence of substitutions of the second kind. If ϕ is a linear substitution of the first kind which satisfies the equation

$$\phi\,\Omega\,\phi = \Omega,$$

then if -1 is a root of the characteristic equation of ϕ, the nullity of any positive integer power of $\phi + \delta$ is even.

That substitutions of the second kind actually exist may be shown by considering the form

$$a\,(x_1 y_1 - x_2 y_2) + b\,x_2 y_1,$$

which is transformed automorphically if we impose upon the x's the substitution whose matrix is

$$\begin{matrix} -1 & 1 \\ 0 & -1 \end{matrix}$$

and upon the y's the transverse substitution; that is, if we put

$$x_1 = -\xi_1 + \xi_2, \qquad x_2 = -\xi_2,$$

and

$$y_1 = -\eta_1, \qquad y_2 = \eta_1 - \eta_2.$$

This linear substitution does not satisfy the preceding conditions, and is therefore of the second kind. It follows that there are one or more bilinear forms for any number of variables which are transformed automorphically in the manner we have considered in this section by linear substitution of the second kind.

By definition, no solution of the second kind of the equation

$$\phi\,\Omega\,\phi = \Omega$$

in an even power of a solution of this equation. But if ϕ is a solution of the second kind, for any positive integer m, we can find a linear substitution satisfying this equation whose $(2m+1)$th power is equal to ϕ. Thus, employing the notation of pages 185, 186, let $\vartheta\,\Omega$ be a polynomial in ϕ such that

$$\phi = e^{\vartheta\,\Omega};$$

and, as above, let

$$\theta_0\,\Omega = \tfrac{1}{2}\,(\vartheta\,\Omega + \Omega\,\vartheta), \qquad \theta_1\,\Omega = \tfrac{1}{2}\,(\vartheta\,\Omega - \Omega\,\vartheta).$$

Then, if

$$\psi = e^{\theta_0\,\Omega + \frac{1}{2m+1}\theta_1\,\Omega},$$

$$\psi\,\Omega\,\psi = \Omega,$$

and

$$\psi^{2m+1} = \phi.$$

Corresponding to any linear substitution ϕ of the second kind satisfying the equation

$$\phi\,\Omega\,\phi = \Omega,$$

can always be found a solution ϕ_ζ of the first kind whose coefficients are rational functions of a parameter ζ, such that, by taking ζ sufficiently small, the coefficients of ϕ_ζ may be made as nearly as we please equal to the corresponding coefficients of ϕ.

§ 3.

If the two sets of variables of the bilinear form

$$(\Omega \between x_1, x_2, \ldots x_n \between y_1, y_2, \ldots y_n),$$

of non-zero determinant, are transformed by a linear substitution whose product is equal to the identical substitution; thus, if

$$(x_1, x_2, \ldots x_n) = (\phi \between \xi_1, \xi_2, \ldots \xi_n),$$

$$(y_1, y_2, \ldots y_n) = (\phi^{-1} \between \eta_1, \eta_2, \ldots \eta_n),$$

we have

$$(\breve{\phi}^{-1}\,\Omega\,\phi \between \xi_1, \xi_2, \ldots \xi_n \between \eta_1, \eta_2, \ldots \eta_n)$$
$$= (\Omega \between x_1, x_2, \ldots x_n \between y_1, y_2, \ldots y_n);$$

and the necessary and sufficient condition that the transformation shall be automorphic is

$$\breve{\phi}^{-1}\,\Omega\,\phi = \Omega.$$

Any one of the class of linear substitutions which satisfy this equation is the mth power of a linear substitution of this class, and can be

generated by the repetition of an infinitesimal substitution of this class. For let $\chi = f(\phi)$ be a polynomial in ϕ such that

$$\phi = e^{\chi}.*$$

Let

$$\vartheta = f(\phi)\,\Omega^{-1},$$

that is,

$$\vartheta\,\Omega = f(\phi);$$

then

$$\breve{\Omega}\,\breve{\vartheta} = f(\breve{\phi}).$$

And since

$$\phi = \Omega^{-1}\,\breve{\phi}\,\Omega,$$

and consequently

$$\breve{\phi} = \breve{\Omega}\,\phi\,\breve{\Omega}^{-1},$$

we have

$$\breve{\Omega}\,\breve{\vartheta} = f(\breve{\Omega}\,\phi\,\breve{\Omega}^{-1}) = \breve{\Omega}\cdot f(\phi)\cdot\breve{\Omega}^{-1}.$$

Therefore

$$\breve{\vartheta}\,\breve{\Omega} = f(\phi) = \vartheta\,\Omega.$$

If now m is any positive integer, and

$$\psi = e^{\frac{1}{m}\vartheta\,\Omega} = e^{\frac{1}{m}\breve{\vartheta}\,\breve{\Omega}},$$

then

$$\breve{\psi} = e^{\frac{1}{m}\breve{\Omega}\,\breve{\vartheta}} = e^{\frac{1}{m}\Omega\,\vartheta},$$

$$\breve{\psi}^{-1} = e^{-\frac{1}{m}\breve{\Omega}\,\breve{\vartheta}} = e^{-\frac{1}{m}\Omega\,\vartheta};$$

and we have identically

$$\breve{\psi}^{-1}\,\Omega\,\psi = e^{-\frac{1}{m}\Omega\,\vartheta}\,\Omega\,e^{\frac{1}{m}\vartheta\,\Omega} = \Omega.$$

We also have

$$\psi^{m} = e^{\vartheta\,\Omega} = \phi.$$

Consequently any linear substitution ϕ which satisfies the equation

$$\breve{\phi}^{-1}\,\Omega\,\phi = \Omega$$

is the mth power for any positive integer m of a linear substitution ψ which also satisfies this equation; and since by taking m sufficiently great we can make ψ as nearly as we please equal to the identical substitution, ϕ can be generated by the repetition of an infinitesimal substitution which also satisfies this equation.

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* See note, page 183.